

# The mixed Hodge structure on complements of complex coordinate subspace arrangements

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**ABSTRACT.** We compute the mixed Hodge structure on the cohomology ring of complements of complex coordinate subspace arrangements. The mixed Hodge structure can be described in terms of the special bigrading on the cohomology ring of complements of complex coordinate subspace arrangements. Originally this bigrading was introduced in the setting of toric topology by V.M. Buchstaber and T.E. Panov.

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## Introduction

A study of topology of coordinate subspace arrangements appears in different areas of mathematics: in toric topology and combinatorial topology [3, 2], in the theory of toric varieties, where complements to coordinate subspace arrangements play the role of homogeneous coordinate spaces [4, 5], in the theory of integral representations of holomorphic functions in several complex variables, where coordinate subspace arrangements play the role of singular sets of integral representations kernels [1, 11].

The universal combinatorial method for the computation of cohomology groups of complements to *arbitrary* subspace arrangements was developed in the book of Goresky and Macpherson [8] (see also [12]), but this method often leads to cumbersome computations. In the study of toric topology, in particular, in works of Buchstaber and Panov [3, 2], the method for the computation of the cohomology of complements to *coordinate* subspace arrangements was developed, this method is simpler than the universal method and allows to get some additional topological information.

The main purpose of this article is to compute the mixed Hodge structure on the cohomology rings of complements to complex coordinate subspace arrangements. We will show that the mixed Hodge structure is described by means of a special bigrading on the cohomology rings of complements to complex coordinate subspace

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arrangements, which was introduced in [3, 2], this bigrading was obtained originally from the combinatorial and topological ideas.

The first section of this paper consists of different facts about topology of complements to complex coordinate subspace arrangements, in the text of this section we follow [3], [2]. Let  $Z_{\mathcal{K}}$  be a complex coordinate subspace arrangement in  $\mathbb{C}^n$ , this arrangement is defined by combinatorics of simplicial complex  $\mathcal{K}$  on the set  $\{1, \dots, n\}$ . In [3], [2], from the topological reasons, the differential bigraded algebra  $R_{\mathcal{K}}$  was introduced ( $R_{\mathcal{K}}$  is determined by combinatorics of  $\mathcal{K}$ ), the ring of cohomology  $H^*(\mathbb{C}^n \setminus Z_{\mathcal{K}})$  is isomorphic to the ring of cohomology  $H^*(R_{\mathcal{K}})$ . Denote by  $H^{-p, 2q}(R_{\mathcal{K}})$  the bigraded cohomology of the algebra  $R_{\mathcal{K}}$ , then

$$H^s(\mathbb{C}^n \setminus Z_{\mathcal{K}}) \simeq \bigoplus_{-p+2q=s} H^{-p, 2q}(R_{\mathcal{K}}).$$

Thus, there is the bigrading on the cohomology ring  $H^*(\mathbb{C}^n \setminus Z)$ .

In the second section we recall some facts and concept from differential topology and algebraic geometry. These facts we use in the last sections.

In the third section the main theorem of this paper is proved. We will show that the bigrading on the cohomology of  $R_{\mathcal{K}}$  and, consequently, the bigrading on the cohomology  $H^*(\mathbb{C}^n \setminus Z_{\mathcal{K}})$  appear naturally from the mixed Hodge structure on cohomology of  $\mathbb{C}^n \setminus Z$ . In particular, denote by  $F^k H^s(\mathbb{C}^n \setminus Z, \mathbb{C})$  a  $k$ -th term of the Hodge filtration on  $H^s(\mathbb{C}^n \setminus Z, \mathbb{C})$ , and denote by  $W_k H^s(\mathbb{C}^n \setminus Z, \mathbb{C})$  a  $k$ -th term of the weight filtration on  $H^s(\mathbb{C}^n \setminus Z, \mathbb{C})$ . Then there is the following theorem.

THEOREM 1.

$$\begin{aligned} F^k H^s(\mathbb{C}^n \setminus Z_{\mathcal{K}}, \mathbb{C}) &\cong \bigoplus_{\substack{q \geq k \\ -p+2q=s}} H^{-p, 2q}(R_{\mathcal{K}}) \otimes \mathbb{C}, \\ W_r H^s(\mathbb{C}^n \setminus Z_{\mathcal{K}}, \mathbb{C}) &\cong \bigoplus_{\substack{2q \leq r \\ -p+2q=s}} H^{-p, 2q}(R_{\mathcal{K}}) \otimes \mathbb{C}. \end{aligned}$$

## 1. General facts on topology of coordinate subspace arrangements

This section consists from different facts about topology of complements of coordinate subspaces arrangements. All statements of this section are taken from [3].

Let  $\mathcal{K}$  be an arbitrary simplicial complex on the set  $[n] = \{1, \dots, n\}$ , i.e. the vertices of  $\mathcal{K}$  are elements of  $[n]$ . An element  $j \in [n]$  is called a *ghost vertex* if  $j$  is not a vertex of  $\mathcal{K}$ . Let  $\sigma = \{i_1, \dots, i_m\}$  be a subset of  $[n]$ , we write  $\sigma \notin \mathcal{K}$  if  $\sigma$  does not define a simplex in  $\mathcal{K}$ . Define the coordinate planes arrangement corresponding to  $\mathcal{K}$

$$Z_{\mathcal{K}} := \bigcup_{\sigma \notin \mathcal{K}} L_{\sigma},$$

where

$$L_{\sigma} = \{z \in \mathbb{C}^n : z_i = 0, i \in \sigma\}.$$

Every arrangement of complex coordinate subspaces in  $\mathbb{C}^n$  can be defined in this way.

Let  $D_{\sigma}^2 \times S_{\gamma}^1$  be the following chain

$$D_{\sigma}^2 \times S_{\gamma}^1 = \{|z_i| \leq 1 : i \in \sigma; |z_j| = 1 : j \in \gamma, z_k = 1 : k \notin \gamma \cup \sigma\},$$

where  $\sigma, \gamma \subseteq [n]$  and  $\sigma \cap \gamma = \emptyset$ . Consider the differential form

$$(1) \quad \frac{dz_I}{z_I} = \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_k}}{z_{i_k}},$$

where  $I \subseteq [n]$ ,  $|I| = k$ ,  $I = \{i_1, \dots, i_k\}$ , and  $i_1 < \cdots < i_k$ .

The orientation of the chain  $D_\sigma^2 \times S_\gamma^1$  is such that the restriction of the form

$$\frac{1}{(\sqrt{-1})^{|\gamma|}} \frac{dz_\gamma}{z_\gamma} \wedge \bigwedge_{j \in \sigma} (\sqrt{-1} dz_j \wedge d\bar{z}_j)$$

to  $D_\sigma^2 \times S_\gamma^1$  is positive. Then the boundary of this chain equals

$$\partial D_\sigma^2 \times S_\gamma^1 = \sum_{i \in \sigma} (-1)^{(i, \gamma \cup i) + 1} D_{\sigma \setminus i}^2 \times S_{\gamma \cup i}^1,$$

where  $(i, \gamma \cup i)$  is the position of  $i$  in the naturally ordered set  $\gamma \cup i$ .

DEFINITION 1. *The topological space*

$$\mathcal{Z}_K = \bigcup_{\sigma \in K} D_\sigma^2 \times S_{[n] \setminus \sigma}^1$$

is called the moment-angle complex.

THEOREM 2 ([3]). *There is a deformation retraction of  $\mathbb{C}^n \setminus Z_K$  onto  $\mathcal{Z}_K$ .*

Let us introduce some formal algebraic notations. They will be used to describe the topology of moment-angle complexes.

DEFINITION 2. *A Stanley-Reisner ring of a simplicial complex  $K$  on the vertex set  $[n]$  is a ring*

$$\mathbb{Z}[K] = \mathbb{Z}[v_1, \dots, v_n] / \mathcal{I}_K,$$

where  $\mathcal{I}_K$  is a homogeneous ideal generated by the monomials  $v_\sigma = \prod_{i \in \sigma} v_i$  such that  $\sigma \notin K$ :

$$\mathcal{I}_K = (v_{i_1} \cdots v_{i_m} : \{i_1, \dots, i_m\} \notin K).$$

Consider the differential bigraded algebra  $(R(K), \delta_R)$ :

$$R_K := \Lambda[u_1, \dots, u_n] \otimes \mathbb{Z}[K] / \mathcal{J},$$

where  $\Lambda[u_1, \dots, u_n]$  is an exterior algebra and  $\mathcal{J}$  is the ideal generated by monomials  $v_i^2, u_i \otimes v_i, i = 1, \dots, n$ . The bidegrees of generators  $v_i, u_i$  of this algebra are equal to

$$\text{bideg } v_i = (0, 2), \text{ bideg } u_i = (-1, 2).$$

The differential  $\delta_R$  is defined on the generators as follows

$$\delta_R u_i = v_i, \delta_R v_i = 0.$$

Denote  $u_I v_J := u_{i_1} \cdots u_{i_q} \otimes v_{j_1} \cdots v_{j_p}$ , where  $I, J \subseteq [n]$ ,  $I = \{i_1, \dots, i_q\}$ ,  $i_1 < \cdots < i_q$ ,  $J = \{j_1, \dots, j_p\}$ , and  $I \cap J = \emptyset$  (we suppose that  $u_\emptyset v_\emptyset = 1$ ).

Consider  $R_K^{-p, 2q}$  the homogeneous component of the algebra  $R_K$  of the bidegree  $(-p, 2q)$ . The differential  $\delta_R$  is compatible with the bigrading, i.e.,  $\delta_R(R_K^{-p, 2q}) \subseteq R_K^{-p+1, 2q}$ . Consider the complex

$$\cdots \xrightarrow{\delta_R} R_K^{-p-1, 2q} \xrightarrow{\delta_R} R_K^{-p, 2q} \xrightarrow{\delta_R} R_K^{-p+1, 2q} \xrightarrow{\delta_R} \cdots,$$

let  $H^{-p,2q}(R_{\mathcal{K}})$  be a cohomology group of this complex. It is clear that the cohomology groups of  $R_{\mathcal{K}}$  are isomorphic to

$$H^s(R_{\mathcal{K}}) = \bigoplus_{-p+2q=s} H^{-p,2q}(R_{\mathcal{K}}).$$

**THEOREM 3 ([3]).** *The cohomology ring  $H^*(\mathbb{C}^n \setminus Z_{\mathcal{K}})$  is isomorphic to the ring  $H^*(R_{\mathcal{K}})$ .*

**REMARK 1.** *The relation between Theorem 3 and the results of Goresky and Macpherson [8] on cohomology of subspace arrangements is described in [3, Ch. 8].*

Let us describe an explicit construction of the isomorphism of Theorem 3. First we construct a cell decomposition of  $Z_{\mathcal{K}}$ . Consider the cell

$$E_{\sigma\gamma} = \{|z_i| < 1 : i \in \sigma; |z_j| = 1, z_j \neq 1 : j \in \gamma; z_k = 1 : k \notin \gamma \cup \sigma\},$$

where  $\sigma, \gamma \subseteq [n]$ , and  $\sigma \cap \gamma = \emptyset$ . The closure of this cell equals  $\overline{E_{\sigma\gamma}} = D_{\sigma}^2 \times S_{\gamma}^1$ . The orientation of  $E_{\sigma\gamma}$  is defined by the orientation of  $D_{\sigma}^2 \times S_{\gamma}^1$ . We obtain the cell decomposition

$$Z_{\mathcal{K}} = \bigcup_{\sigma \in \mathcal{K}, \gamma \subseteq [n] \setminus \sigma} E_{\sigma\gamma}.$$

Let  $C_*(Z_{\mathcal{K}})$  be the cellular chains group of this cell decomposition, then by  $C^*(Z_{\mathcal{K}})$  denote the cellular cochains group. Let  $E'_{\sigma\gamma}$  be a cocell dual to the cell  $E_{\sigma\gamma}$ , i.e.,  $E'_{\sigma\gamma}$  is a linear function on  $C_*(Z_{\mathcal{K}})$  such that  $\langle E'_{\sigma\gamma}, E_{\sigma'\gamma'} \rangle = \delta_{\sigma'\gamma'}^{\sigma\gamma}$  (the Kronecker delta).

**PROPOSITION 1 ([3]).** *The linear map  $\phi : R_{\mathcal{K}} \rightarrow C^*(Z_{\mathcal{K}})$ ,  $\phi(v_{\sigma}u_{\gamma}) = E'_{\sigma\gamma}$  is an isomorphism of differential bigraded modules. In particular, there is the isomorphism  $H^*(R_{\mathcal{K}}) \xrightarrow{\phi} H^*(Z_{\mathcal{K}})$ .*

From the structure of the cell decomposition of  $Z_{\mathcal{K}}$  and Theorem 2 it follows that every cycle  $\Gamma \in H_s(\mathbb{C}^n \setminus Z_{\mathcal{K}})$  has a representative of the form

$$(2) \quad \Gamma = \sum_{-p+2q=s} \Gamma_{-p,2q},$$

where  $\Gamma_{-p,2q}$  is a cycle of the form

$$(3) \quad \Gamma_{-p,2q} = \sum_{\substack{|\sigma|=q-p \\ |\gamma|=p}} C_{\sigma\gamma} \cdot D_{\sigma}^2 \times S_{\gamma}^1, C_{\sigma\gamma} \in \mathbb{Z}.$$

Let  $H_{-p,2q}(\mathbb{C}^n \setminus Z_{\mathcal{K}})$  be a group generated by all cycles of the form (3). Obviously, we have

$$H_s(\mathbb{C}^n \setminus Z_{\mathcal{K}}) = \bigoplus_{-p+2q=s} H_{-p,2q}(\mathbb{C}^n \setminus Z_{\mathcal{K}}).$$

**PROPOSITION 2.** *The pairing between the vector spaces  $H_{-p,2q}(\mathbb{C}^n \setminus Z_{\mathcal{K}}, \mathbb{R}) \subset H_{-p+2q}(\mathbb{C}^n \setminus Z_{\mathcal{K}}, \mathbb{R})$  and  $\phi(H^{-p',2q'}(R_{\mathcal{K}} \otimes \mathbb{R})) \subset H^{-p'+2q'}(\mathbb{C}^n \setminus Z_{\mathcal{K}}, \mathbb{R})$  is nondegenerate if  $p = p'$ ,  $q = q'$  and equals to zero otherwise.*

*Proof.* Suppose  $p' \neq p$  and  $q' \neq q$  then it follows from Proposition 1 that  $\langle \Gamma_{-p,2q}, \phi(\omega^{-p',2q'}) \rangle = 0$  for every  $\Gamma_{-p,2q} \in H_{-p,2q}(\mathbb{C}^n \setminus Z_{\mathcal{K}})$  and every  $\omega^{-p',2q'} \in$

$H^{-p', 2q'}(R_{\mathcal{K}})$ . Hence the pairing between  $\Gamma_{-p, 2q}$  and  $\phi(\omega^{-p', 2q'})$  can be nonzero only if  $p' = p, q' = q$ . Since the pairing between

$$H_s(\mathbb{C}^n \setminus Z_{\mathcal{K}}, \mathbb{R}) = \bigoplus_{-p+2q=s} H_{-p, 2q}(\mathbb{C}^n \setminus Z_{\mathcal{K}}, \mathbb{R})$$

and

$$H^s(\mathbb{C}^n \setminus Z_{\mathcal{K}}, \mathbb{R}) = \bigoplus_{-p+2q=s} \phi(H^{-p, 2q}(R_{\mathcal{K}} \otimes \mathbb{R}))$$

is nondegenerate, we obtain the statement of the Proposition.

## 2. Mixed Hodge structures and resolvents of cycles

Here we recall some facts about mixed Hodge structures. For references, see [6], [9], [10], [13].

DEFINITION 3. *Let  $H$  be a finite-dimensional vector space over  $\mathbb{Q}$ . A pure Hodge structure of weight  $s$  on  $H$  is a decreasing filtration  $F$  on  $H_{\mathbb{C}} = H \otimes \mathbb{C}$ , such that*

$$F^p H_{\mathbb{C}} \cap \overline{F^q H_{\mathbb{C}}} = 0$$

*for any  $p + q = s + 1$ . The filtration  $F$  is called the Hodge filtration.*

DEFINITION 4. *Let  $H$  be a finite-dimensional vector space over  $\mathbb{Q}$ . A mixed Hodge structure on  $H$  consists, by definition, of the following:*

- (1) *An increasing (weight) filtration  $W$  on  $H$*
- (2) *A decreasing (Hodge) filtration  $F$  on  $H = H \otimes \mathbb{C}$ , satisfying the following condition: the filtration  $F$  induces a pure Hodge structure of weight  $s$  on  $\text{Gr}_s^W H = W_s H \otimes \mathbb{C} / W_{s-1} H \otimes \mathbb{C}$ .*

Cohomology groups of quasi-projective varieties admit a natural mixed Hodge structure. In particular,

PROPOSITION 3. *The weight filtration on the cohomology  $H^s(X, \mathbb{Q})$  of a smooth variety  $X$  has the form*

$$0 = W_{s-1} \subset W_s \subset \cdots \subset W_{2s} = H^s(X, \mathbb{Q}).$$

In this article we will use methods of differential topology, so we will consider the weight filtration on  $H^s(X, \mathbb{C})$ , not on  $H^s(X, \mathbb{Q})$ . Now we recall the construction of the mixed Hodge structure on cohomology groups of a smooth complex algebraic variety.

Let  $X$  be a smooth complex algebraic variety of dimension  $n$ . A proper compactification of a variety  $X$  is an open embedding  $j : X \hookrightarrow \overline{X}$  into a complete smooth algebraic variety  $\overline{X}$ , such that  $\overline{X} \setminus X = V$  is a smooth normal crossing divisor. According to Hironaka's theorem, a proper compactification always exists.

Let the divisor  $V$  be defined by equations  $z_1 \cdots z_k = 0$  in the neighborhood  $U \subset \overline{X}$ , where  $z_i$  are local coordinates in  $U$ .

DEFINITION 5. *The sheaf*

$$\Omega_X^m(\log V) = \bigwedge^m (\Omega_X^1(\log V))$$

is called the sheaf of holomorphic  $m$ -forms on  $\overline{X}$  with logarithmic poles along  $V$ , where  $\Omega_{\overline{X}}^1$  is the locally free  $\mathcal{O}_{\overline{X}}$ -module, generated over  $U$  by the differentials

$$\frac{dz_1}{z_1}, \dots, \frac{dz_k}{z_k}, dz_{k+1}, \dots, dz_n.$$

In other words, the sections of the sheaf  $\Omega_{\overline{X}}^m(\log V)$  in the neighborhood  $U$  are  $m$ -forms

$$\omega \wedge \frac{dz_I}{z_I},$$

where  $\omega$  is a holomorphic form on  $U$ ,  $I = \{i_1, \dots, i_p\} \subset \{1, \dots, k\}$ . Consider the following sheafs

$$\begin{aligned} \mathcal{E}_{\overline{X}}^{p,q}(\log V) &= \Omega_{\overline{X}}^p(\log V) \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{E}_{\overline{X}}^{0,q}, \\ \mathcal{E}_{\overline{X}}^s(\log V) &= \bigoplus_{p+q=s} \mathcal{E}_{\overline{X}}^{p,q}(\log V). \end{aligned}$$

One can show

$$H^s(X, \mathbb{C}) = H_d^s(\Gamma(\overline{X}, \mathcal{E}_{\overline{X}}^s(\log V))),$$

where  $\Gamma(\overline{X}, \mathcal{E}_{\overline{X}}^s(\log V))$  is the space of global sections of  $\mathcal{E}_{\overline{X}}^s(\log V)$  and  $H_d^s(\Gamma(\overline{X}, \mathcal{E}_{\overline{X}}^s(\log V)))$  is the cohomology with respect to the external derivative  $d$ .

Let us define the increasing weight filtration  $W$  on the sheaf  $\Omega_{\overline{X}}^m(\log V)$ , by setting

$$W_k \Omega_{\overline{X}}^m(\log V) = \Omega_{\overline{X}}^k(\log V) \wedge \Omega_{\overline{X}}^{m-k},$$

similarly on  $\mathcal{E}_{\overline{X}}^{p,q}(\log V)$  and  $\mathcal{E}_{\overline{X}}^s(\log V)$ . Define the weight filtration on cohomology as follows

$$W_{k+s} H^s(X, \mathbb{C}) = H_d^s(\Gamma(\overline{X}, W_k \mathcal{E}_{\overline{X}}^s(\log V))).$$

Consider the decreasing Hodge filtration

$$F^k \mathcal{E}_{\overline{X}}^s(\log V) = \bigoplus_{p \geq k} \mathcal{E}_{\overline{X}}^{p,s-p}(\log V)$$

on the sheaf  $\mathcal{E}_{\overline{X}}^s(\log V)$ . There is the induced Hodge filtration on the cohomology

$$F^k H^s(X, \mathbb{C}) = H_d^s(\Gamma(\overline{X}, F^k \mathcal{E}_{\overline{X}}^s(\log V))).$$

**PROPOSITION 4.** *These two filtration define the mixed Hodge structure on  $H^s(X, \mathbb{C})$ .*

Now we are going to develop some technical methods, we will use them to prove the main result of the paper. The main idea of what we are going to do is to define the weight filtration and the Hodge filtration on some kind of Čech-de Rham complex. Probably, the constructions below are well-known, but we don't know references.

Let  $D_\alpha, \alpha \in \mathcal{A}$  be a finite set of divisors in  $\overline{X}$  such that  $\bigcup_{\alpha \in \mathcal{A}} D_\alpha \cup V$  is a smooth normal crossing divisor and  $\bigcap_{\alpha \in \mathcal{A}} D_\alpha = \emptyset$ . This set of divisors defines the open cover  $\mathcal{U} = \{\mathcal{U}_\alpha = \overline{X} \setminus D_\alpha\}_{\alpha \in \mathcal{A}}$  of  $\overline{X}$ . Also, it defines the open cover  $\mathcal{U}_X = \{\mathcal{U}_{X_\alpha} = X \setminus D_\alpha\}_{\alpha \in \mathcal{A}}$  of  $X$ .

Consider the Čech complex  $(C^*(\mathcal{U}, \mathcal{E}^r(\log V)), \delta)$  of the cover  $\mathcal{U}$  with coefficients in the sheaf  $\mathcal{E}^r(\log V)$

$$\dots \xrightarrow{\delta} C^{t-1}(\mathcal{U}, \mathcal{E}^r(\log V)) \xrightarrow{\delta} C^t(\mathcal{U}, \mathcal{E}^r(\log V)) \xrightarrow{\delta} C^{t+1}(\mathcal{U}, \mathcal{E}^r(\log V)) \rightarrow \dots$$

Let  $(C^*(\mathcal{U}, \mathcal{E}_{\log}^r), \delta)$  be a subcomplex of  $(C^*(\mathcal{U}, \mathcal{E}^r(\log V)), \delta)$  such that for any  $\omega \in C^t(\mathcal{U}, \mathcal{E}_{\log}^r)$  the element  $(\omega)_{\alpha_0, \dots, \alpha_t}$  is an element of  $\Gamma(\overline{X}, \mathcal{E}_{\overline{X}}^r(\log V \cup D_{\alpha_0} \cup \dots \cup D_{\alpha_t}))$ . There is the de Rham differential  $d : C^t(\mathcal{U}, \mathcal{E}_{\log}^r) \rightarrow C^t(\mathcal{U}, \mathcal{E}_{\log}^{r+1})$ ,

$$(d\omega)_{\alpha_0, \dots, \alpha_t} = d(\omega)_{\alpha_0, \dots, \alpha_t}.$$

The groups  $C^t(\mathcal{U}, \mathcal{E}_{\log}^r)$  with the differentials  $d$  and  $\delta$  form a double complex, the associated single complex is denoted by

$$K^s(\mathcal{U}, \mathcal{E}_{\log}) = \bigoplus_{r+t=s} C^t(\mathcal{U}, \mathcal{E}_{\log}^r).$$

Consider the operator  $D = (-1)^r \delta + d$  on  $C^t(\mathcal{U}, \mathcal{E}_{\log}^r)$ , this operator defines the differential of the complex  $K^s(\mathcal{U}, \mathcal{E}_{\log})$ . The Hodge filtration  $F$  and the weight filtration  $W$  are defined naturally on the complexes  $K^s(\mathcal{U}, \mathcal{E}_{\log})$  and  $C^t(\mathcal{U}, \mathcal{E}_{\log}^r)$ .

Consider a linear map  $\varepsilon : \mathcal{E}_{\overline{X}}^s(\log V) \rightarrow C^0(\mathcal{U}, \mathcal{E}_{\log}^s)$  such that  $\varepsilon(\omega)_\alpha = \omega|_{\mathcal{U}_\alpha}$ ,  $\alpha \in \mathcal{A}$ . The map  $\varepsilon : \mathcal{E}_{\overline{X}}^s(\log V) \rightarrow C^0(\mathcal{U}, \mathcal{E}_{\log}^s)$  induces a map from  $\mathcal{E}_{\overline{X}}^s(\log V)$  to  $K^s(\mathcal{U}, \mathcal{E}_{\log})$ , we will denote the latter map by the same symbol  $\varepsilon$ .

Let  $\rho_\alpha, \alpha \in \mathcal{A}$  be a partition of unity subordinated to the open cover  $\mathcal{U}$ , i.e. a set of real nonnegative  $C^\infty$ -functions on  $\overline{X}$  such that  $\sum_{\alpha \in \mathcal{A}} \rho_\alpha \equiv 1$  and  $\text{supp}(\rho_\alpha) \subset \mathcal{U}_\alpha$ . Using the partition of unity we define a homotopy operator  $T : C^t(\mathcal{U}, \mathcal{E}_{\log}^r) \rightarrow C^{t-1}(\mathcal{U}, \mathcal{E}_{\log}^r)$ :

$$(T\omega)_{i_0, \dots, i_{t-1}} = \sum_{\alpha \in \mathcal{A}} \rho_\alpha \omega_{\alpha, i_0, \dots, i_{t-1}},$$

here  $\omega$  is an element of  $C^t(\mathcal{U}, \mathcal{E}_{\log}^s)$ . It is easy to check that

$$T\delta + \delta T = \text{Id}.$$

PROPOSITION 5. *The map  $\varepsilon : \mathcal{E}_{\overline{X}}^s(\log V) \rightarrow K^*(\mathcal{U}, \mathcal{E}_{\log})$  is a quasi-isomorphism of complexes, the induced isomorphism*

$$H^*(X, \mathbb{C}) \cong H^*(\mathcal{E}_{\overline{X}}^*(\log V), d) \xrightarrow{\cong} H^*(K^*(\mathcal{U}, \mathcal{E}_{\log}), D)$$

*respects the Hodge filtration and the weight filtration.*

*Proof.* Consider the sequence

$$0 \rightarrow \mathcal{E}_{\overline{X}}^s(\log V) \xrightarrow{\varepsilon} C^0(\mathcal{U}, \mathcal{E}_{\log}^s) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{E}_{\log}^s) \xrightarrow{\delta} \dots,$$

because there is the homotopy operator  $T$ , this sequence is exact. Therefore, by standard arguments,  $\varepsilon$  defines a quasi-isomorphism of complexes. Moreover, the homotopy operator is compatible with the weight filtration and the Hodge filtration, in other word, we have  $T(W_k C^t(\mathcal{U}, \mathcal{E}_{\log}^s)) \subset W_k C^{t-1}(\mathcal{U}, \mathcal{E}_{\log}^s)$  and  $T(F^k C^t(\mathcal{U}, \mathcal{E}_{\log}^s)) \subset F^k C^{t-1}(\mathcal{U}, \mathcal{E}_{\log}^s)$ . Hence the sequence

$$0 \rightarrow W_k \mathcal{E}_{\overline{X}}^s(\log V) \xrightarrow{\varepsilon} W_k C^0(\mathcal{U}, \mathcal{E}_{\log}^s) \xrightarrow{\delta} W_k C^1(\mathcal{U}, \mathcal{E}_{\log}^s) \xrightarrow{\delta} \dots,$$

$$0 \rightarrow F^k \mathcal{E}_{\overline{X}}^s(\log V) \xrightarrow{\varepsilon} F^k C^0(\mathcal{U}, \mathcal{E}_{\log}^s) \xrightarrow{\delta} F^k C^1(\mathcal{U}, \mathcal{E}_{\log}^s) \xrightarrow{\delta} \dots$$

are exact. Hence, the induced filtrations on the cohomology groups of  $K^*(\mathcal{U}, \mathcal{E}_{\log})$  coincide with the filtrations on  $H^*(\mathcal{E}_{\overline{X}}^*(\log V), d)$ .

From now until the end of this section we will follow the ideas the paper [7]. The main purpose of this part is to give definitions of a  $\mathcal{U}_X$ -chain and of a  $\mathcal{U}_X$ -resolvent of a cycle, and to describe their properties.

DEFINITION 6. A  $\mathcal{U}_X$ -chain of degree  $t$  and of dimension  $s$  on the variety  $X$  is an alternating function  $\Gamma$  from the set of indexes  $\mathcal{A}^{t+1}$  to the group of  $s$ -dimensional singular chains of  $X$  such that  $\Gamma$  is nonzero at a finite number of points from  $\mathcal{A}^{t+1}$  and

$$\text{supp}(\Gamma_{i_0, \dots, i_t}) \subset \mathcal{U}_{X_{i_0}} \cap \dots \cap \mathcal{U}_{X_{i_t}},$$

for every  $(i_0, \dots, i_t) \in \mathcal{A}^{t+1}$ , where  $\text{supp}(\Gamma_{i_0, \dots, i_t})$  is the support of the chain  $\Gamma_{i_0, \dots, i_t}$ .

Let  $C_{t,s}(\mathcal{U}_X)$  be an additive group of  $\mathcal{U}_X$ -chains of degree  $t$  and of dimension  $s$  on the variety  $X$ . Define maps  $\delta' : C_{t,s}(\mathcal{U}_X) \rightarrow C_{t-1,s}(\mathcal{U}_X)$

$$(\delta'\Gamma)_{i_0, \dots, i_{t-1}} = \sum_{i \in \mathcal{A}} \Gamma_{i, i_0, \dots, i_{t-1}},$$

and  $\partial : C_{t,s}(\mathcal{U}_X) \rightarrow C_{t,s-1}(\mathcal{U}_X)$

$$(\partial\Gamma)_{i_0, \dots, i_t} = \partial(\Gamma)_{i_0, \dots, i_t},$$

i.e., the operator  $\partial$  is a boundary operator on each chain  $\Gamma_{i_0, \dots, i_t}$ . Obviously, we have  $\partial\partial = 0$  and  $\delta'\delta' = 0$ . The groups  $C_{t,s}(\mathcal{U}_X)$ ,  $t, s \geq 0$  with the differentials  $\delta', \partial$  form a double complex. Let us define the map  $\varepsilon : C_{0,s}(\mathcal{U}_X) \rightarrow C_s(X)$  in the following way

$$\varepsilon(\Gamma) = \sum_{i \in \mathcal{A}} \Gamma_i,$$

where  $C_s(X)$  is the group of the singular chains of dimension  $s$  on  $X$ .

Now we will construct a pairing between elements of  $C_{t,s}(\mathcal{U}_X)$  and  $C^t(\mathcal{U}, \mathcal{E}_{\log}^s)$ . Suppose  $\Gamma \in C_{t,s}(\mathcal{U}_X)$  and  $\omega \in C^t(\mathcal{U}, \mathcal{E}_{\log}^s)$ , then

$$\langle \omega, \Gamma \rangle = \frac{1}{(t+1)!} \sum_{(i_0, \dots, i_t) \in \mathcal{A}^{t+1}} \int_{\Gamma_{i_0, \dots, i_t}} \omega_{i_0, \dots, i_t}.$$

There are the following relations for the pairing:

$$\begin{aligned} \langle \omega^{t,s}, \partial\Gamma_{t,s+1} \rangle &= \langle d\omega^{t,s}, \Gamma_{t,s+1} \rangle, \\ \langle \delta\omega^{t,s}, \Gamma_{t+1,s} \rangle &= \langle \omega^{t,s}, \delta'\Gamma_{t+1,s} \rangle, \\ \int_{\varepsilon(\Gamma_{0,s})} \omega^s &= \langle \varepsilon\omega^s, \Gamma_{0,s} \rangle, \end{aligned}$$

where  $\omega^{t,s} \in C^t(\mathcal{E}^s, \mathcal{U})$ ,  $\omega^s \in \mathcal{E}^s(X)$ , and  $\Gamma_{t,s} \in C_{t,s}(\mathcal{U})$ .

DEFINITION 7. Let  $\Gamma$  be a singular cycle of dimension  $s$  on  $X$ , then a  $\mathcal{U}_X$ -resolvent of length  $k$  of the cycle  $\Gamma$  is a collection of  $\mathcal{U}_X$ -chains  $\Gamma^i \in C_{i,s-i}(\mathcal{U}_X)$ ,  $0 \leq i \leq k$  such that  $\Gamma = \varepsilon\Gamma^0$  and  $\partial\Gamma^i = (-1)^{s-i}\delta'\Gamma^{i+1}$ .

PROPOSITION 6. Given an  $s$ -dimensional cycle  $\Gamma$ , a closed differential form  $\omega$  of degree  $s$  on  $X$ , a  $\mathcal{U}_X$ -resolvent  $\Gamma^0, \dots, \Gamma^k$  of the cycle  $\Gamma$ , and a cocycle  $\tilde{\omega} \in K^s(\mathcal{U}, \mathcal{E}_{\log})$ ,  $\tilde{\omega} = \sum_{i \leq k} \tilde{\omega}^{i,s-i}$ ,  $\tilde{\omega}^{i,s-i} \in C^i(\mathcal{U}, \mathcal{E}_{\log}^{s-i})$ , and the cocycle  $\varepsilon\omega$  equals  $\tilde{\omega}$  in  $H^s(K^*(\mathcal{U}, \mathcal{E}_{\log}), D)$ , then

$$\int_{\Gamma} \omega = \sum_{i \leq k} \langle \tilde{\omega}^{i,s-i}, \Gamma^i \rangle.$$

This proposition follows directly from the properties of the pairing.



REMARK 2. We considered  $\mathcal{U}_X$ -chains on the manifold  $X$  with some special cover  $\mathcal{U}_X$  and the pairing with elements of  $C^t(\mathcal{U}, \mathcal{E}_{\log}^s)$ , because it is what we need in the sequel. However, one can consider the  $\mathcal{U}_X$ -chains for an arbitrary cover of a smooth manifold  $X$  and the pairing of this  $\mathcal{U}_X$ -chains with the Čech cochains  $C^t(\mathcal{U}_X, \mathcal{E}^s)$ , in this case the results above are also true.

### 3. The Mixed Hodge structure on cohomology of complements to coordinate subspace arrangements

In this section we compute the Mixed Hodge structure on the cohomology ring  $H^*(\mathbb{C}^n \setminus Z_{\mathcal{K}}, \mathbb{C})$ . It follows from Theorem 3 and Proposition 1 that there is the isomorphism  $H^*(\mathbb{C}^n \setminus Z_{\mathcal{K}}, \mathbb{C}) \stackrel{\phi}{\cong} H^*(R_{\mathcal{K}} \otimes \mathbb{C})$ .

THEOREM 4. Let  $H^{-p,2q}(R_{\mathcal{K}})$  be the bigraded cohomology group of the complex  $R_{\mathcal{K}}^{-p,2q}$ , then there is the isomorphisms

$$F^k H^s(\mathbb{C}^n \setminus Z_{\mathcal{K}}, \mathbb{C}) \stackrel{\phi}{\cong} \bigoplus_{\substack{q \geq k \\ -p+2q=s}} H^{-p,2q}(R_{\mathcal{K}} \otimes \mathbb{C}),$$

$$W_r H^s(\mathbb{C}^n \setminus Z_{\mathcal{K}}, \mathbb{C}) \stackrel{\phi}{\cong} \bigoplus_{\substack{2q \leq r \\ -p+2q=s}} H^{-p,2q}(R_{\mathcal{K}} \otimes \mathbb{C}).$$

*Proof.* Let us construct a proper compactification of  $\mathbb{C}^n \setminus Z_{\mathcal{K}}$ . Consider the standard embedding  $\mathbb{C}^n \setminus Z_{\mathcal{K}} \hookrightarrow \mathbb{C}^n \hookrightarrow \mathbb{CP}^n$ . Then

$$\mathbb{C}^n \setminus Z_{\mathcal{K}} = \mathbb{CP}^n \setminus (\mathbb{CP}_{\infty}^{n-1} \cup \bigcup_{\sigma \notin \mathcal{K}} \overline{L_{\sigma}}),$$

where  $\overline{L_{\sigma}}$  is the closure of the complex plane

$$L_{\sigma} = \{z \in \mathbb{C}^n : z_i = 0, i \in \sigma\},$$

and  $\mathbb{CP}_{\infty}^{n-1}$  is the hyperplane at infinity. Making a sequence of blowups along irreducible components of  $\mathbb{CP}_{\infty}^{n-1} \cup \bigcup_{\sigma \notin \mathcal{K}} \overline{L_{\sigma}}$  we get some proper compactification of  $\mathbb{C}^n \setminus Z_{\mathcal{K}}$ , let us denote this compactification by  $\overline{X}$ . Let  $\tilde{L}_i$  be a proper preimage of a closure of  $L_i = \{z \in \mathbb{C}^n : z_i = 0\}$  in  $\overline{X}$ . Consider

$$\mathcal{U}_{\sigma} = \overline{X} \setminus \bigcup_{i \notin \sigma} \tilde{L}_i,$$

then  $\mathcal{U}_{\mathcal{K}} = \{\mathcal{U}_{\sigma}\}_{\sigma \in \mathcal{K}}$  is the open cover of  $\overline{X}$ , the restriction of  $\mathcal{U}_{\mathcal{K}}$  to  $\mathbb{C}^n \setminus Z_{\mathcal{K}}$  is the open cover of  $\mathbb{C}^n \setminus Z_{\mathcal{K}}$ . Consider  $\tilde{\mathcal{U}}_{\sigma} = \mathcal{U}_{\sigma} \cap (\mathbb{C}^n \setminus Z_{\mathcal{K}})$ ,  $\tilde{\mathcal{U}}_{\sigma}$  is isomorphic to  $\mathbb{C}^{|\sigma|} \times (\mathbb{C}^*)^{n-|\sigma|}$ .

Consider the logarithmic Čech-de Rham double complex  $(C^t(\mathcal{U}_{\mathcal{K}}, \mathcal{E}_{\log}^r), d, \delta)$  and the associated single complex  $(K^s(\mathcal{U}_{\mathcal{K}}, \mathcal{E}_{\log}), D)$ . From Proposition 5 the cohomology of  $H^s(K^*(\mathcal{U}_{\mathcal{K}}, \mathcal{E}_{\log}), D)$  is isomorphic to the de Rham cohomology group  $H^s(\mathbb{C}^n \setminus Z_{\mathcal{K}}, \mathbb{C})$ . The weight filtration on the booth cohomology groups coincide, also the same fact is true for the the Hodge filtration. Define a subcomplex  $(M^{r,t}(\mathcal{U}_{\mathcal{K}}), d, \delta)$  of  $(C^t(\mathcal{U}, \mathcal{E}_{\log}^r), d, \delta)$ , the elements of this complex are following

$$(\omega)_{\alpha_0 \dots \alpha_t} = \sum_{\substack{|I|=r \\ I \subset [n] \setminus (\alpha_0 \cap \dots \cap \alpha_t)}} C_{\alpha_0 \dots \alpha_t}^I \frac{dz_I}{z_I},$$

where  $C_{\alpha_0 \dots \alpha_t}^I \in \mathbb{C}$  and  $\alpha_0, \dots, \alpha_t \in \mathcal{K}$ , it is easy to see that  $\frac{dz_I}{z_I}$  are indeed a logarithmic form in the proper compactification. Denote by  $j : M^{r,t}(\mathcal{U}_{\mathcal{K}}) \hookrightarrow C^t(\mathcal{U}, \mathcal{E}_{\log}^r)$  the inclusion of the subcomplex. The associated single complex is denoted  $M^s(\mathcal{U}_{\mathcal{K}})$ . Notice that differential  $d$  acts trivially on the complex  $M^{r,t}(\mathcal{U}_{\mathcal{K}})$ , hence the cohomology  $H^s(M^\bullet(\mathcal{U}_{\mathcal{K}}), D)$  is isomorphic to  $\bigoplus_{r+t=s} H^t(M^{r,\bullet}(\mathcal{U}_{\mathcal{K}}), \delta)$ , where  $H^t(M^{r,\bullet}(\mathcal{U}_{\mathcal{K}}), \delta)$  is the cohomology of the complex

$$\dots \rightarrow M^{r,t-1}(\mathcal{U}_{\mathcal{K}}) \rightarrow M^{r,t}(\mathcal{U}_{\mathcal{K}}) \rightarrow M^{r,t+1}(\mathcal{U}_{\mathcal{K}}) \rightarrow \dots$$

LEMMA 1. *The inclusion map  $j : M^s(\mathcal{U}_{\mathcal{K}}) \hookrightarrow K^s(\mathcal{U}_{\mathcal{K}}, \mathcal{E}_{\log})$  is the quasi-isomorphism.*

*Proof of the Lemma.* We will show that for any element  $\varphi \in K^s(\mathcal{U}_{\mathcal{K}}, \mathcal{E}_{\log})$  such that  $D\varphi = 0$ , there are  $\psi \in M^s(\mathcal{U}_{\mathcal{K}})$  and  $\omega \in K^{s-1}(\mathcal{U}_{\mathcal{K}}, \mathcal{E}_{\log})$  such that  $D\psi = 0$ ,  $\varphi = \psi + D\omega$ . This will prove the lemma. First, notice that any cocycle in  $K^s(\mathcal{U}_{\mathcal{K}}, \mathcal{E}_{\log})$  is cohomologous to a cocycle from  $C^0(\mathcal{U}_{\mathcal{K}}, \mathcal{E}_{\log}^s)$ , indeed, by Proposition 5 the map  $\varepsilon : \mathcal{E}_{\overline{X}}^s(\log V) \rightarrow C^0(\mathcal{U}_{\mathcal{K}}, \mathcal{E}_{\log}^s) \subset K^s(\mathcal{U}_{\mathcal{K}}, \mathcal{E}_{\log})$  is a quasi-isomorphism of complexes. Let  $\varphi$  be a cocycle from  $C^0(\mathcal{U}_{\mathcal{K}}, \mathcal{E}_{\log}^s)$ . Let us prove that for any  $k \geq -1$  there are  $\psi^0, \dots, \psi^k, \psi^i \in M^{s-i,i}(\mathcal{U}_{\mathcal{K}})$ ,  $D\psi^i = 0$ ,  $\omega^0, \dots, \omega^k, \omega^i \in C^i(\mathcal{U}_{\mathcal{K}}, \mathcal{E}_{\log}^{s-1-i})$ , and  $\varphi^k \in C^{k+1}(\mathcal{U}_{\mathcal{K}}, \mathcal{E}_{\log}^{s-k-1})$ ,  $D\varphi^k = 0$  such that

$$\varphi = \sum_{i=0}^k \psi^i + D \sum_{i=0}^k \omega^i + \varphi^k.$$

We will construct inductively such cycle decomposition. The base of induction:  $k = -1$ , this case is trivial,  $\varphi = \varphi^{-1}$ .

Suppose that the decomposition

$$\varphi = \sum_{i=0}^k \psi^i + D \sum_{i=0}^k \omega^i + \varphi^k$$

is already constructed for given  $k$ .

Since  $D\varphi^k = 0$ , the form  $\varphi_{\sigma_0, \dots, \sigma_{k+2}}^k, \sigma_0, \dots, \sigma_{k+2} \in \mathcal{K}$  is a closed  $(s-k-1)$ -form on  $\tilde{\mathcal{U}}_{\sigma_0 \cap \dots \cap \sigma_{k+2}}$ . There is a unique decomposition

$$\varphi_{\sigma_0, \dots, \sigma_{k+2}}^k = \sum_{\substack{|I|=s-k-1 \\ I \subset [n] \setminus (\sigma_0 \cap \dots \cap \sigma_{k+2})}} C_{\sigma_0, \dots, \sigma_{k+2}}^I \frac{dz_I}{z_I} + d\omega_{\sigma_0, \dots, \sigma_{k+2}},$$

indeed, differential forms  $\frac{dz_I}{z_I}, |I| = s-k-1, I \subset [n] \setminus (\sigma_0 \cap \dots \cap \sigma_{k+2})$  are the basis of  $H^{s-k-1}(\tilde{\mathcal{U}}_{\sigma_0 \cap \dots \cap \sigma_{k+2}})$ . Taking  $\omega_{\sigma_0, \dots, \sigma_{k+2}}^k = -\omega_{\sigma_0, \dots, \sigma_{k+2}}$ ,  $\varphi^{k+1} = (-1)^{s-k-1} \delta \omega^k$ , and

$$\psi_{\sigma_0, \dots, \sigma_{k+2}}^{k+1} = \sum_{\substack{|I|=s-k-1 \\ I \subset [n] \setminus (\sigma_0 \cap \dots \cap \sigma_{k+2})}} C_{\sigma_0, \dots, \sigma_{k+2}}^I \frac{dz_I}{z_I},$$

we obtain

$$\varphi = \sum_{i=0}^{k+1} \psi^i + D \sum_{i=0}^{k+1} \omega^i + \varphi^{k+1}.$$

Let us check that  $D\psi^{k+1} = 0, D\varphi^{k+1} = 0$ . From definition we have

$$D\psi^{k+1} + D\varphi^{k+1} = 0, d\psi^{k+1} = 0, \delta\varphi^{k+1} = 0,$$

therefore,  $(-1)^{s-k-1}\delta\psi^{k+1} = d\psi^{k+1}$ ,  $(\delta\psi^{k+1})_{\sigma_0, \dots, \sigma_{k+3}}$  is a linear combination of differential forms  $\frac{dz_I}{z_I}$  on  $\tilde{\mathcal{U}}_{\sigma_0 \cap \dots \cap \sigma_{k+3}} \cong \mathbb{C}^{|\sigma_0 \cap \dots \cap \sigma_{k+3}|} \times (\mathbb{C}^*)^{n-|\sigma_0 \cap \dots \cap \sigma_{k+3}|}$ , this form is exact if and only if it is identically zero. Thus,  $(-1)^{s-k-1}\delta\psi^{k+1} = d\psi^{k+1} = 0$ . The induction step is proved.

Taking  $k = s$  we have

$$\varphi = \sum_{i=0}^s \psi^i + D \sum_{i=0}^s \omega^i + 0,$$

for  $\psi = \sum_{i=0}^s \psi^i$  and  $\omega = \sum_{i=0}^s \omega^i$ , we have

$$\varphi = \psi + D\omega,$$

this proves the Lemma.  $\square$

LEMMA 2. *The Mixed Hodge Structure on  $H^s(K^*(\mathcal{U}_{\mathcal{K}}, \mathcal{E}_{\log}), D) \cong H^s(M^*(\mathcal{U}_{\mathcal{K}}), D)$  is the following*

$$F^k H^s(M^*(\mathcal{U}_{\mathcal{K}}), D) = \bigoplus_{\substack{r \geq k \\ r+t=s}} H^t(M^{r, \bullet}(\mathcal{U}_{\mathcal{K}}), \delta),$$

$$W_k H^s(M^*(\mathcal{U}_{\mathcal{K}}), D) = \bigoplus_{\substack{2r \leq k \\ r+t=s}} H^t(M^{r, \bullet}(\mathcal{U}_{\mathcal{K}}), \delta).$$

*Proof of the Lemma.* The complex  $(M^{r, t}(\mathcal{U}_{\mathcal{K}}), \delta)$  is naturally isomorphic to the complex  $(C^t(\mathcal{U}_{\mathcal{K}}, H^r(\bullet)), \delta)$ . An element  $\psi \in C^t(\mathcal{U}_{\mathcal{K}}, H^r(\bullet))$  is a cochain with coefficients in cohomology, i.e.  $\psi_{\sigma_0, \dots, \sigma_t} \in H^r(\tilde{\mathcal{U}}_{\sigma_0 \cap \dots \cap \sigma_t}, \mathbb{C})$ . There is the natural Mixed Hodge Structure on  $H^r(\tilde{\mathcal{U}}_{\sigma_0 \cap \dots \cap \sigma_t}, \mathbb{C})$ :

$$H^*(\tilde{\mathcal{U}}_{\sigma_0 \cap \dots \cap \sigma_t}, \mathbb{C}) \simeq \bigwedge^* \left[ \frac{dz_i}{z_i} : i \notin \sigma_0 \cap \dots \cap \sigma_t \right],$$

$$F^k H^*(\tilde{\mathcal{U}}_{\sigma_0 \cap \dots \cap \sigma_t}, \mathbb{C}) \simeq \bigoplus_{r \geq k} \bigwedge^r \left[ \frac{dz_i}{z_i} : i \notin \sigma_0 \cap \dots \cap \sigma_t \right],$$

$$W_k H^*(\tilde{\mathcal{U}}_{\sigma_0 \cap \dots \cap \sigma_t}, \mathbb{C}) \simeq \bigoplus_{2r \leq k} \bigwedge^r \left[ \frac{dz_i}{z_i} : i \notin \sigma_0 \cap \dots \cap \sigma_t \right].$$

By functoriality we obtain the Mixed Hodge Structure on  $H^s(M^*(\mathcal{U}_{\mathcal{K}}), D)$ . This gives us the statement of the Lemma.  $\square$

LEMMA 3. *Let*

$$\Gamma_{-p, 2q} = \sum_{\substack{|\sigma|=q-p \\ |\gamma|=p}} C^{\sigma\gamma} \cdot D_{\sigma}^2 \times S_{\gamma}^1$$

be a cycle in  $\mathbb{C}^n \setminus Z_{\mathcal{K}}$ . Then there is a  $\mathcal{U}_{\mathcal{K}}$ -resolvent  $\Gamma_{-p, 2q}^0, \dots, \Gamma_{-p, 2q}^{q-p}$  on length  $q-p$ , where  $\Gamma_{-p, 2q}^k$  is  $\mathcal{U}_{\mathcal{K}}$ -chain of dimension  $2q-p-k$  and of degree  $k$  of the following form

$$(\Gamma_{-p, 2q}^k)_{\alpha_0, \dots, \alpha_k} = \sum_{\substack{|\sigma|=q-p-k \\ |\gamma|=p+k}} C_{\alpha_0 \dots \alpha_k}^{\sigma\gamma} \cdot D_{\sigma}^2 \times S_{\gamma}^1.$$

*Proof of the Lemma.* We will use the induction on the length  $k$  of the resolvent. We going to construct the resolvent of the special form

$$(\Gamma_{-p,2q}^k)_{\sigma_k, \sigma_{k-1}, \dots, \sigma_0} = \sum_{|\gamma|=p+k} C_{\sigma_k \dots \sigma_0}^{\sigma_k \gamma} \cdot D_{\sigma_k}^2 \times S_{\gamma}^1,$$

for  $|\sigma_j| = q - p - j$ ,  $\sigma_j \subset \sigma_t, j > t, t, j = 0, \dots, k$ , and  $(\Gamma_{-p,2q}^k)_{\alpha_0, \dots, \alpha_k} = 0$  for any other indexes  $\alpha_0, \dots, \alpha_k$ .

The base of induction: define  $(\Gamma_{-p,2q}^0)_{\sigma_0} = \sum_{|\gamma|=p} C_{\sigma_0}^{\sigma_0 \gamma} \cdot D_{\sigma_0}^2 \times S_{\gamma}^1$  with  $|\sigma_0| = q - p$ , and  $(\Gamma_{-p,2q}^0)_{\alpha} = 0$  for any other indexes  $\alpha$ . We get

$$\Gamma_{-p,2q} = \sum_{\substack{|\sigma_0|=q-p \\ |\gamma|=p}} C_{\sigma_0}^{\sigma_0 \gamma} \cdot D_{\sigma_0}^2 \times S_{\gamma}^1 = \sum_{\sigma \in \mathcal{K}} (\Gamma_{-p,2q}^0)_{\sigma} = \varepsilon' \Gamma_{-p,2q}^0,$$

therefore  $\Gamma_{-p,2q}^0$  is the resolvent of length 0.

Suppose that the resolvent  $\Gamma_{-p,2q}^0, \dots, \Gamma_{-p,2q}^k$  of length  $k$  is already constructed. Recall that  $(i, \gamma)$  is the position of  $i$  in the naturally ordered set  $\gamma \cup i$ . Define

$$(\Gamma_{-p,2q}^{k+1})_{\sigma_k \setminus i, \sigma_k \dots \sigma_0} = (-1)^{2q-p-k} \sum_{|\gamma|=p+k} (-1)^{(i, \gamma)} C_{\sigma_k \dots \sigma_0}^{\sigma_k \gamma} D_{\sigma_k \setminus i}^2 \times S_{\gamma \cup i}^1$$

for  $i \in \sigma_k$ ,  $|\sigma_j| = q - p - j$ ,  $\sigma_{j+1} \subset \sigma_j$ , and

$$(\Gamma_{-p,2q}^{k+1})_{\alpha_0, \dots, \alpha_{k+1}} = 0$$

for any other indexes  $\alpha_0, \dots, \alpha_{k+1}$ . Let us show that  $\Gamma_{-p,2q}^0, \dots, \Gamma_{-p,2q}^{k+1}$  is a resolvent of length  $k+1$ . We have

$$\begin{aligned} (-1)^{2q-p-k} (\delta' \Gamma_{-p,2q}^{k+1})_{\sigma_k \dots \sigma_0} &= \sum_{i \in \sigma_k} (\Gamma_{-p,2q}^{k+1})_{\sigma_k \setminus i, \sigma_k \dots \sigma_0} = \\ &= \sum_{\substack{i \in \sigma_k \\ |\gamma|=p+k}} (-1)^{(i, \gamma)} C_{\sigma_k \dots \sigma_0}^{\sigma_k \gamma} D_{\sigma_k \setminus i}^2 \times S_{\gamma \cup i}^1 = \sum_{|\gamma|=p+k} C_{\sigma_k \dots \sigma_0}^{\sigma_k \gamma} \partial D_{\sigma_k}^2 \times S_{\gamma}^1 = \\ &= (\partial \Gamma_{-p,2q}^k)_{\sigma_k \dots \sigma_0}. \end{aligned}$$

For any indexes  $\alpha_0, \dots, \alpha_k$  different from  $\sigma_k, \dots, \sigma_{m+1}, \sigma_{m-1}, \dots, \sigma_0$ ,  $0 \leq m \leq k$ , directly from definition of the chain  $\Gamma_{-p,2q}^k, \Gamma_{-p,2q}^{k+1}$ , we get

$$(\partial \Gamma_{-p,2q}^k)_{\alpha_0, \dots, \alpha_k} = (-1)^{2q-p-k} (\delta' \Gamma_{-p,2q}^{k+1})_{\alpha_0, \dots, \alpha_k} = 0.$$

Consider the last case  $\sigma_k \setminus i, \sigma_k, \dots, \sigma_{m+1}, \sigma_{m-1}, \dots, \sigma_0$ , for  $0 \leq m \leq k$ . Since by the induction hypothesis  $\Gamma_{-p,2q}^0 \dots \Gamma_{-p,2q}^k$  is a resolvent, we get  $(-1)^{2q-p-k+1} \delta' \Gamma_{-p,2q}^k = \partial \Gamma_{-p,2q}^{k-1}$ , hence we have  $\delta' \partial \Gamma_{-p,2q}^k = 0$ , and

$$\begin{aligned} (\delta' \partial \Gamma_{-p,2q}^k)_{\sigma_k \dots \sigma_{m+1} \sigma_{m-1} \dots \sigma_0} &= \\ &= \sum_{\substack{\sigma_{m+1} \subset \sigma_m \subset \sigma_{m-1} \\ |\sigma_m|=q-p-m}} \sum_{|\gamma|=p+k} \sum_{i \in \sigma_k} (-1)^{(i, \gamma)} C_{\sigma_k \dots \sigma_0}^{\sigma_k \gamma} \cdot D_{\sigma_k \setminus i}^2 \times S_{\gamma \cup i}^1 = 0, \end{aligned}$$

Therefore, for a fixed  $i \in \sigma_k$ , we get

$$\sum_{\substack{\sigma_{m+1} \subset \sigma_m \subset \sigma_{m-1} \\ |\sigma_m|=q-p-m}} \sum_{|\gamma|=p+k} (-1)^{(i, \gamma)} C_{\sigma_k \dots \sigma_0}^{\sigma_k \gamma} \cdot D_{\sigma_k \setminus i}^2 \times S_{\gamma \cup i}^1 = 0.$$

On the other side,

$$\begin{aligned} (\delta' \Gamma_{-p,2q}^{k+1})_{\sigma_k \setminus i, \sigma_k \dots \sigma_{m+1} \sigma_{m-1} \dots \sigma_0} &= \\ &= \sum_{\substack{\sigma_{m+1} \subset \sigma_m \subset \sigma_{m-1} \\ |\sigma_m| = q = p-m}} \sum_{|\gamma| = p+k} (-1)^{(i,\gamma)} C_{\sigma_k \dots \sigma_0}^{\sigma_k \gamma} \cdot D_{\sigma_k \setminus i}^2 \times S_{\gamma \cup i}^1, \end{aligned}$$

hence  $(\delta' \Gamma_{-p,2q}^{k+1})_{\sigma_k \setminus i, \sigma_k \dots \sigma_{m+1} \sigma_{m-1} \dots \sigma_0} = 0$ . We have shown that  $\partial \Gamma_{-p,2q}^k = (-1)^{2q-p-k} \delta' \Gamma_{-p,2q}^{k+1}$ .  $\square$

Recall that by Proposition 5 the map  $\varepsilon : H^*(\mathbb{C}^n \setminus Z_{\mathcal{K}}, \mathbb{C}) \rightarrow H^*(K^*(\mathcal{U}_{\mathcal{K}}, \mathcal{E}_{\log}))$  is an isomorphism.

LEMMA 4. *Let*

$$\Gamma_{-p,2q} = \sum_{\substack{|\sigma| = q-p \\ |\gamma| = p}} C^{\sigma \gamma} \cdot D_{\sigma}^2 \times S_{\gamma}^1$$

be a cycle in  $\mathbb{C}^n \setminus Z_{\mathcal{K}}$ , and let  $\psi \in M^{r,t} \subseteq K^{r+t}(\mathcal{U}_{\mathcal{K}}, \mathcal{E}_{\log})$  be a cocycle. Then

$$\int_{\Gamma_{-p,2q}} \varepsilon^{-1} \psi = 0,$$

for any  $p, q, r, t$  such that  $r \neq q$  or  $t \neq q - p$ .

*Proof of the Lemma.* We may assume that  $2q - p = r + t = s$ , otherwise  $\dim \Gamma_{-p,2q} \neq \deg \varepsilon^{-1} \psi$  and the integral is automatically zero. By Lemma 3 we have the resolvent  $\Gamma_{-p,2q}^0, \dots, \Gamma_{-p,2q}^{q-p}$  of  $\Gamma_{-p,2q}$ .

The first case is  $r > q$ . By Proposition 6 we have

$$\int_{\Gamma_{-p,2q}} \varepsilon^{-1} \psi = \langle \psi, \Gamma_{-p,2q}^t \rangle,$$

where

$$(\Gamma_{-p,2q}^t)_{\alpha_0, \dots, \alpha_t} = \sum_{\substack{|\sigma| = q-p-t \\ |\gamma| = p+t}} C_{\alpha_0 \dots \alpha_t}^{\sigma \gamma} \cdot D_{\sigma}^2 \times S_{\gamma}^1.$$

Notice, that  $q - p - t > 0$ , the pairing  $\langle \psi, \Gamma_{-p,2q}^t \rangle$  is a sum of integrals of the differential forms  $\frac{dz_I}{z_I}$  over chains  $D_{\sigma}^2 \times S_{\gamma}^1, |\sigma| \neq 0$ , direct computations show that all this integrals are equal to zero. So, we obtain

$$\int_{\Gamma_{-p,2q}} \varepsilon^{-1} \psi = 0.$$

The second case  $r < q$ . There are cochains  $\omega^{r,s-r-1}, \dots, \omega^{q-1,s-q}, \omega^{i,j} \in C^j(\mathcal{U}_{\mathcal{K}}, \mathcal{E}_{\log}^i)$  such that the cocycle

$$\varphi = \psi + D \sum_{i=r}^{q-1} \omega^{i,s-i-1}$$

is an element of  $C^{s-q}(\mathcal{U}_{\mathcal{K}}, \mathcal{E}_{\log}^q)$ . By Proposition 6 we have

$$\int_{\Gamma_{-p,2q}} \varepsilon^{-1} \psi = \langle \varphi, \Gamma_{-p,2q}^{q-p} \rangle,$$

where

$$(\Gamma_{-p,2q}^{q-p})_{\alpha_0, \dots, \alpha_t} = \sum_{|\gamma| = q} C_{\alpha_0 \dots \alpha_{q-p}}^{\gamma} \cdot S_{\gamma}^1.$$

Notice, that the differential forms  $\varphi_{\sigma_0, \dots, \sigma_{s-q}} = d(\omega_{\sigma_0, \dots, \sigma_{s-q}}^{q-1, s-q})$  are exact. Thus, the pairing  $\langle \varphi, \Gamma_{-p, 2q}^{q-p} \rangle$  is a sum of integrals of exact differential forms over cycles  $S_\gamma^1$ , therefore, all such integrals are equal to zero. So, we obtain

$$\int_{\Gamma_{-p, 2q}} \varepsilon^{-1} \psi = 0.$$

The Lemma is proved.  $\square$

By Proposition 2 the pairing between the vector spaces  $H_{-p, 2q}(\mathbb{C}^n \setminus Z_K, \mathbb{C}) \subset H_{-p+2q}(\mathbb{C}^n \setminus Z_K, \mathbb{C})$  and  $\phi(H^{-p', 2q'}(R_K \otimes \mathbb{C})) \subset H^{-p'+2q'}(\mathbb{C}^n \setminus Z_K, \mathbb{C})$  is non-degenerate if  $p = p'$ ,  $q = q'$  and equals to zero otherwise. In the other hand  $H^*(\mathbb{C}^n \setminus Z_K, \mathbb{C}) \xrightarrow{\varepsilon} H^*(K^*(\mathcal{U}_K, \mathcal{E}_{\log})) \simeq H^s(M^\bullet(\mathcal{U}_K), D) \simeq \bigoplus_{r+t=s} H^t(M^{r, \bullet}(\mathcal{U}_K), \delta)$ . By Lemma 4 we see that the pairing between  $H_{-p, 2q}(\mathbb{C}^n \setminus Z_K, \mathbb{C}) \subset H_{-p+2q}(\mathbb{C}^n \setminus Z_K, \mathbb{C})$  and  $H^t(M^{r, \bullet}(\mathcal{U}_K), \delta)$  is zero if  $r \neq q$  or  $t \neq q - p$ . The pairing between  $H_{-p+2q}(\mathbb{C}^n \setminus Z_K, \mathbb{C})$  and  $H^{-p+2q}(M^\bullet(\mathcal{U}_K), D)$  is nondegenerate, therefore, the pairing between  $H_{-p, 2q}(\mathbb{C}^n \setminus Z_K, \mathbb{C})$  and  $H^q - p(M^{q, \bullet}(\mathcal{U}_K), \delta)$  is non degenerate, and these spaces are dual. Hence, we have the natural isomorphism

$$H^{q-p}(M^{q, \bullet}(\mathcal{U}_K), \delta) \xrightarrow{\phi \circ \varepsilon} H^{-p, 2q}(R_K \otimes \mathbb{C}),$$

using this isomorphism and Lemma 2 we obtain the statement of the Theorem.  $\square$

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